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ON THE THEORY OF REGULAR PIECEWISE-HOMOGENEOUS STRUCTURES WITH PIEZOCERAMIC MATRICES*

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A piecewise-homogeneous medium consisting of a piezoceramic matrix bonded by a doubly-periodic system of anisotropic fibres, dielectrics, is considered. The electroelasticity boundary value problems occurring here reduce to a system of Fredholm integral equations of the second kind whose solvability is proved. Concepts of mean mechanical and electrical quantities are introduced from energy considerations, between which a relationship is given by the equations of state of the structure macromodels. The algorithm constructed is realized numerically. Results are presented of computations of the average elastic, electrical, and piezoelectrical properties of the medium as a function of the cell microstructure.

Models of elastic linearly-reinforced composite materials with isotropic and anisotropic components were examined for example, in /1–3/. A survey of the results in the area of electroelasticity boundary value problems can be found in /4/.

1. Formulation of the problem.

We consider a transversely isotropic piezoelectric medium (a crystal of the hexagonal 6 mm system, PZT-4, PZT-5, etc. piezoceramic, prepolarized along the z axis), reinforced by a doubly-periodic system of identical anisotropic fibres along the y axis, referred to the crystallographic xyz axes. The fibre transverse cross-section is a simply-connected domain bounded by a simple closed curve l with curvature satisfying the Hölder condition /5/. The fundamental periods of the structure are denoted by ω_1 and $\omega_2\,(\mathrm{Im}\,(\omega_2/\omega_1)>0)$ the domain occupied by the matrix by D, and the domain occupied by the fibre in the unit cell Π_0 by D_0.

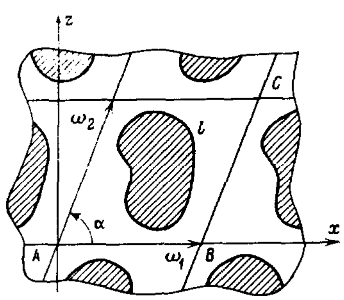

Fig.1

For such an idealization in the plane of the transverse section we obtain an infinitely connected domain that is invariant under the group of translations $T(z)=z+P$, where P is the complex period (Fig.1). We shall assume the mean components of the mechanical stress tensor $\langle\sigma_x\rangle$, $\langle\tau_{xz}\rangle$, $\langle\sigma_z\rangle$ and the electrical intensity vector $\langle E_x\rangle$, $\langle E_z\rangle$ act in the structure.

We will construct a model of a regular piezoceramic medium under the following additional assumptions: a) all fibres have identical physicomechanical properties and possess a plane of elastic symmetry perpendicular to the y axis; b) conditions hold for ideal electrical and mechanical contact between the fibre and the matrix. Under these conditions the fields of the mechanical stresses, the induction and intensity vectors of the electrical field possess the same symmetry group as does the domain D.

The mechanical and electrical quantities in the matrix are defined by the formulas

$$U=2\,\mathrm{Re}\sum_{k=1}^{3}p_k\Phi_k(z_k),\quad W=2\,\mathrm{Re}\sum_{k=1}^{3}q_k\Phi_k(z_k) \tag{1.1}$$

$$\sigma_x=2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k\mu_k^2\Phi_k'(z_k),\quad \tau_{xz}=-2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k\mu_k\Phi_k'(z_k)$$

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$$\sigma_z = 2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k\Phi_k'(z_k),\quad z_k = x + \mu_k z \quad (k = 1, 2, 3)$$

$$E_x = 2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\Phi_k'(z_k),\quad E_z = 2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\mu_k\Phi_k'(z_k)$$

$$D_x = 2\,\mathrm{Re}\sum_{k=1}^{3}r_k\mu_k\Phi_k'(z_k),\quad D_z = -2\,\mathrm{Re}\sum_{k=1}^{3}r_k\Phi_k'(z_k)$$

Here $\Phi_k(z_k)$ and $\Phi_k'(z_k) = d\Phi_k(z_k)/dz_k$ are analytic functions of the complex variable z_k while the remaining quantities in (1.1) are defined in /6/.

The mechanical and electric fields in the fibre are not interrelated, they are defined by the relationships (the quantities therein are defined in /7, 8/)

$$U = 2\,\mathrm{Re}\sum_{k=4}^{5}p_k\Phi_k(z_k),\quad W = 2\,\mathrm{Re}\sum_{k=4}^{5}q_k\Phi_k(z_k) \tag{1.2}$$

$$\sigma_x = 2\,\mathrm{Re}\sum_{k=4}^{5}\mu_k^2\Phi_k'(z_k),\quad \tau_{xz} = -2\,\mathrm{Re}\sum_{k=4}^{5}\mu_k\Phi_k'(z_k)$$

$$\sigma_z = 2\,\mathrm{Re}\sum_{k=4}^{5}\Phi_k'(z_k),\quad \Delta = \varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2$$

$$E_x = 2\,\mathrm{Re}\,\Phi_6'(z_6),\quad E_z = 2\,\mathrm{Re}\,[\mu_6\Phi_6'(z_6)]$$

$$D_x = 2\,\mathrm{Re}\,[(\varepsilon_{11} + \mu_6\varepsilon_{13})\,\Phi_6'(z_6)]$$

$$D_z = 2\,\mathrm{Re}\,[(\varepsilon_{13} + \mu_6\varepsilon_{33})\,\Phi_6'(z_6)]$$

The conditions for ideal mechanical and electrical contact between the fibres and the matrix consist of continuity of the mechanical stress and displacement vectors as well as of the tangential component of the electrical field intensity vector and the normal component of its induction vector through the interface of the media. By virtue of (1.1) and (1.2), these connection conditions have the form

$$2\,\mathrm{Re}\sum_{k=1}^{3}A_{nk}\Phi_k(t_k) = 2\,\mathrm{Re}\sum_{k=4}^{6}B_{nk}\Phi_k(t_k) \tag{1.3}$$

$$A_{1k} = p_k,\quad A_{2k} = q_k,\quad A_{3k} = \gamma_k\mu_k,\quad A_{4k} = \gamma_k,\quad A_{5k} = \lambda_k,$$

$$A_{6k} = r_k \quad (k = 1, 2, 3);\quad B_{1k} = p_k,\quad B_{2k} = q_k,\quad B_{3k} = \mu_k,$$

$$B_{4k} = 1,\quad B_{5k} = B_{6k} = 0 \quad (k = 4, 5),$$

$$B_{16} = B_{26} = B_{36} = B_{46} = 0,\quad B_{56} = 1,\quad B_{66} = -i\sqrt{\Delta};\quad t_k =$$

$$\mathrm{Re}\,t + \mu_k\,\mathrm{Im}\,t\ (k = 1, 2, \ldots, 6)$$

2. Mean values of the mechanical and electrical quantities. It follows from the formulation of the problem that the mechanical displacements U, W and the electric field potential φ are quasiperiodic functions; consequently, the mean values of the mechanical strains $\langle e_x\rangle$, $\langle\gamma_{xz}\rangle$, $\langle\varepsilon_z\rangle$, the rigid rotation of the cell $\langle\omega_{xz}\rangle$ and the electric field intensity $\langle E_x\rangle$, $\langle E_z\rangle$ are determined accurately from the relationships

$$\Delta_1 U = \omega_1\langle e_x\rangle,\quad \Delta_2 U = h\langle e_x\rangle + \frac{1}{2}H(\langle\gamma_{xz}\rangle + \langle\omega_{xz}\rangle) \tag{2.1}$$

$$\Delta_1 W = \frac{1}{2}\omega_1(\langle\gamma_{xz}\rangle - \langle\omega_{xz}\rangle),\quad \Delta_2 W = H\langle\varepsilon_z\rangle +$$

$$\frac{1}{2}h(\langle\gamma_{xz}\rangle - \langle\omega_{xz}\rangle)$$

$$\Delta_1\varphi = -\omega_1\langle E_x\rangle,\quad \Delta_2\varphi = -h\langle E_x\rangle - H\langle E_z\rangle$$

$$(h = \mathrm{Re}\,\omega_2,\quad H = \mathrm{Im}\,\omega_2;\quad \Delta_m\varkappa = \varkappa(z + \omega_m) - \varkappa(z),\ m = 1, 2)$$

Furthermore, if $\langle X_n\rangle$, $\langle Z_n\rangle$ are components of the principal mechanical force vector acting on some face of the cell, while $\langle D_n\rangle$ is the total electric induction flux vector through this face (Fig.1), the mean values of the mechanical forces $\langle\sigma_x\rangle$, $\langle\tau_{xz}\rangle$, $\langle\sigma_z\rangle$ and the electric induction vector $\langle D_x\rangle$, $\langle D_z\rangle$ are naturally introduced as follows

On the face AB

$$\langle X_n\rangle = \omega_1\langle\tau_{xz}\rangle,\quad \langle Z_n\rangle = \omega_1\langle\sigma_z\rangle,\quad \langle D_n\rangle = \omega_1\langle D_z\rangle \tag{2.2}$$

On the face BC

$$\langle X_n\rangle = |\omega_2|(\langle\sigma_x\rangle\sin\alpha - \langle\tau_{xz}\rangle\cos\alpha) \tag{2.3}$$

$$\langle Z_n\rangle = |\omega_2|(\langle\tau_{xz}\rangle\sin\alpha - \langle\sigma_z\rangle\cos\alpha),$$

$$\langle D_n\rangle = |\omega_2|(\langle D_x\rangle\sin\alpha - \langle D_z\rangle\cos\alpha)$$

The internal energy of the unit cell Π_0 can be converted to the form

$$W_0=\iint\limits_{\Pi_0} W_V\,dx\,dz=\frac{1}{2}\Big[\int\limits_{AB}(X_n\Delta_1U+Z_n\Delta_1W-D_n\Delta_1\varphi)\,ds+ \int\limits_{BC}(X_n\Delta_2U+Z_n\Delta_2W-D_n\Delta_2\varphi)\,ds\Big] \tag{2.4}$$

Here X_n, Z_n, D_n are components of the mechanical stress vector and the normal component of the electric induction vector on the contour of the cell Π_0 and W_V is the internal energy density of the unit cell /9/.

By virtue of (2.2) and (2.3) we obtain finally (F is the area of the unit cell)

$$W_0=\frac{1}{2}F[\langle\varepsilon_x\rangle\langle\sigma_x\rangle+\langle\gamma_{xz}\rangle\langle\tau_{xz}\rangle+\langle\varepsilon_z\rangle\langle\sigma_z\rangle+ \langle D_x\rangle\langle E_x\rangle+\langle D_z\rangle\langle E_z\rangle] \tag{2.5}$$

The correctness of the mean components introduced for the mechanical and electrical quantities follows from the energy equality (2.5). For a unique determination of the states of stress and strain in the structure, as well as the electric intensity and induction, it is obviously necessary and sufficient to specify any of the following four possible systems of mean quantities (1, 3, 5, 7, 9), (1, 3, 5, 8, 10), (2, 4, 6, 7, 9), (2, 4, 6, 8, 10), where the numbers correspond to the numerical order of the quantities $\langle\varepsilon_x\rangle, \ldots, \langle E_z\rangle$ in (2.5).

3. The integral equations of the boundary value problem. We represent the desired analytic functions $\Phi_k(z_k)$ in the form

$$\Phi_k(z_k)=\frac{1}{2\pi i}\int\limits_l \zeta_k(t_k-z_k)\,\omega_k(t)\,dt_k+R_kz_k,\ z_k\in D^{(k)}\quad (k=1,2,3) \tag{3.1}$$

$$\Phi_k(z_k)=\frac{1}{2\pi i}\int\limits_l \zeta_k(t_k-z_k)\sum_{j=1}^{3}[l_{kj}\omega_j(t)+l_{kj}^*\overline{\omega_j(t)}]\,dt_k,\ z_k\in D_0^{(k)}\quad (k=4,5,6)$$

Here $\zeta_k(t_k-z_k)$ is the Weierstrass zeta function constructed in the periods $\omega_{1k}=\omega_1$, $\omega_{2k}=h+\mu_kH$, $D^{(k)}$ and D_0^k are affine images of the domains D and D_0 respectively, $\omega_k(t)$ $(k=1,2,3)$ are the desired functions, and the constants R_k are determined from the conditions for the given mean mechanical stresses and the electric field intensity vector to exist in the structure. The direction of integration in (3.1) is clockwise.

The quantities l_{kj}, l_{kj}^* are determined from the system

$$\sum_{k=4}^{6}(B_{nk}l_{kj}-\bar B_{nk}\overline{l_{kj}^*})=A_{nj}\quad (n=1,2,\ldots,6;\ j=1,2,3) \tag{3.2}$$

Subsituting the limit values of the function (1.3) into the boundary conditions (3.1), we arrive at the system of integral equations

$$2\,\mathrm{Re}\sum_{j=1}^{3}\Big[D_{nj}\omega_j(t_0)+\int\limits_l G_{nj}(t,t_0)\,\omega_j(t)\,ds\Big]=\varphi_n(x_0,z_0)\quad (n=1,2,\ldots,6) \tag{3.3}$$

$$D_{nj}=\sum_{k=4}^{6}B_{nk}l_{kj},\quad G_{nj}(t,t_0)=H_{nj}(t,t_0)+\overline{H_{nj}^*(t,t_0)}$$

$$H_{nj}(t,t_0)=\frac{1}{2\pi i}\sum_{k=4}^{6}B_{nk}\zeta_k(t_k-t_{k0})\,l_{kj}\frac{dt_k}{ds}+ \left\{\frac{f_{nj}^{\circ}}{F}-\frac{A_{nj}}{2\pi i}\Big[t_{j0}\frac{\delta_{1j}}{\omega_1}+\zeta_k(t_j-t_{j0})\Big]\right\}\frac{dt_j}{ds}$$

$$H_{nj}^*(t,t_0)=\frac{1}{2\pi i}\sum_{k=4}^{6}B_{nk}\zeta_k(t_k-t_{k0})\,l_{kj}^*\frac{dt_k}{ds}$$

$$\mathring{f}_{1k}=x_0(a_{14}\gamma_k\mu_k-a_{23}\lambda_k)+{}^1\!/_2z_0p_k$$
$$\mathring{f}_{2k}=-{}^1\!/_2x_0p_k+z_0[{}^1\!/_2(a_{12}-S_{44})\gamma_k\mu_k-(a_{21}-d_{15})\lambda_k]$$
$$\mathring{f}_{3k}=z_0\gamma_k\mu_k,\quad \mathring{f}_{4k}=0,\quad \mathring{f}_{5k}=z_0\lambda_k$$
$$\mathring{f}_{6k}=x_0(a_{23}\gamma_k\mu_k-a_{22}\lambda_k)$$
$$\varphi_1^{\circ}=x_0[a_{14}\langle\sigma_x\rangle+{}^1\!/_2(a_{12}-S_{44})\langle\sigma_z\rangle-a_{23}\langle E_z\rangle]+ z_0({}^1\!/_2S_{44}\langle\tau_{xz}\rangle+{}^1\!/_2d_{15}\langle E_x\rangle)$$
$$\varphi_2^{\circ}=x_0({}^1\!/_2S_{44}\langle\tau_{xz}\rangle+{}^1\!/_2d_{15}\langle E_x\rangle)+ z_0[{}^1\!/_2(a_{12}-S_{44})\langle\sigma_x\rangle+a_{10}\langle\sigma_z\rangle-(a_{21}-d_{15})\langle E_z\rangle]$$
$$\varphi_3^{\circ}=-x_0\langle\tau_{xz}\rangle+z_0\langle\sigma_x\rangle,\quad \varphi_4^{\circ}=x_0\langle\sigma_z\rangle-z_0\langle\tau_{xz}\rangle$$

$$\varphi_5{}^\circ = x_0 \langle E_x\rangle + z_0 \langle E_z\rangle, \quad \varphi_6{}^\circ = x_0 [a_{23} \langle \sigma_x\rangle + (a_{21} - d_{15}) \langle \sigma_z\rangle - a_{22} \langle E_z\rangle] + z_0 (d_{15} \langle \tau_{xz}\rangle + a_{20} \langle E_x\rangle) \quad (\varkappa^\circ = \varkappa (x_0, z_0))$$

Taking (3.2) into account, it can be shown that system (3.2) is a Fredholm system /10/.

4. The uniqueness Theorem. The internal energy density W_V is a positive-definite quadratic form of the components of the mechanical and electrical quantities. For instance, we have for PZT-5 piezoceramic

$$W_V = 0{,}03\cdot 10^{-12}[(-11\sigma_x + 14\sigma_z + 317E_z)^2 + 236\sigma_x{}^2 + 49\sigma_z{}^2 + 133\,338E_x{}^2 + 117\,145E_z{}^2 + (43\tau_{xz} + 526E_x)^2] \tag{4.1}$$

We consider two solutions of the boundary value problem (1.3) that correspond to the very same mean quantities $\langle\sigma_x\rangle, \langle\tau_{xz}\rangle, \langle\sigma_z\rangle, \langle E_x\rangle, \langle E_z\rangle$. The difference between these solutions will again be a solution of this same boundary value problem for

$$\langle\sigma_x\rangle = \langle\tau_{xz}\rangle = \langle\sigma_z\rangle = \langle E_x\rangle = \langle E_z\rangle = 0 \tag{4.2}$$

By virtue of (2.5) and the positive-definiteness of the internal energy functional, we conclude that the homogeneous boundary value problem (1.3) yields zero mechanical stress and electrical field intensity fields in each component of the structure.

We have the following system to determine the functions $\Phi_k (z_k)$ $(k = 1, 2, 3)$:

$$2 \operatorname{Re} \sum_{k=1}^{3} \alpha_{nk}\Phi_k'(z_k) = \delta_n{}^6\Omega \quad (n = 1, 2, \ldots, 6) \tag{4.3}$$

$$\alpha_{1k} = \gamma_k\mu_k{}^2, \quad \alpha_{2k} = \gamma_k\mu_k, \quad \alpha_{3k} = \gamma_k, \quad \alpha_{4k} = \lambda_k\mu_k$$

$$\alpha_{5k} = \lambda_k, \quad \alpha_{6k} = p_k\mu_k - q_k$$

(Ω is the rigid rotation, and $\delta^6{}_n$ is the Kronecker delta). We hence find that

$$\Phi_k' (z_k) = \Omega g_k \ (k = 1, 2, 3) \tag{4.4}$$

$$2 \operatorname{Re} \sum_{k=1}^{3} g_k\mu_k^{n-1} = \frac{\delta_n{}^6}{2} (a_{14}a_{22} - a_{23}{}^2)^{-1} \quad (n = 1, 2, \ldots, 6)$$

Integrating (4.4), we obtain (d_k are complex constants)

$$\Phi_k (z_k) = \Omega g_k z_k + d_k \ (k = 1, 2, 3) \tag{4.5}$$

The functions describing the appropriate homogeneous boundary value problem in the fibre are found analogously

$$\Phi_k (z_k) = \Omega_0 g_k z_k + d_k \ (k = 4, 5, 6); \ g_6 = 0 \tag{4.6}$$

The connections

$$\Omega = \Omega_0, \quad 2 \operatorname{Re} \sum_{k=1}^{3} A_{nk}d_k = 2 \operatorname{Re} \sum_{k=4}^{6} B_{nk}d_k \quad (n = 1, 2, \ldots, 6) \tag{4.7}$$

result from the boundary conditions (1.3).

5. Solvability of the integral Eqs.(3.3). We consider the homogeneous system corresponding to the system (3.3). The equations $\varphi_n{}^0 (x_0, z_0) = 0$ $(n = 1, 2, \ldots, 6)$ and the relationships (4.2) are obviously equivalent.

We shall later ascribe a zero subscript to the solution of the homogeneous system and all the functions and functionals corresponding to this solution.

Equating the functions (3.1) to the corresponding functions from (4.5) and setting the mean rotation of the cell equal to zero, we find

$$R_{k0} = 0, \ \Omega = \Omega_0 = 0 \tag{5.1}$$

We introduce the following functions into consideration

$$\frac{1}{i} \mathrm{X}_k (z_k) = \frac{1}{2\pi i} \int_l \zeta_k (t_k - z_k)\, \omega_{k0} (t)\, dt_k, \ z_k \in D_0^{(k)} \quad (k = 1, 2, 3) \tag{5.2}$$

$$i\Theta_k (z_k) = \frac{1}{2\pi i} \int_l \zeta_k (t_k - z_k) \sum_{j=1}^{3} [l_{kj}\omega_{j0} (t) + l_{kj}^*\overline{\omega_{j0} (t)}]\, dt_k$$

$$z_k \in D^{(k)} \quad (k = 4, 5, 6)$$

The difference between the limit values of the corresponding functions in (3.1) and (5.2) and l yields

$$\omega_{k0} (t) = i\dot{\mathrm{X}}_k (t_k) + d_k \quad (k = 1, 2, 3) \tag{5.3}$$

$$\sum_{j=1}^{3} [l_{kj}\omega_{j0} (t) + l_{kj}^*\overline{\omega_{j0} (t)}] = i\Theta_k (t_k) - d_k \quad (k = 4, 5, 6)$$

It follows from the first equation in (5.3) that $\omega_{k0} (t)$ are boundary values of functions regular in $D_0^{(k)}$. Then on the basis of (3.1), (4.7) and (5.1) we obtain

$$d_k = 0 \quad (k = 1, 2, \ldots, 6) \tag{5.4}$$

Eliminating the function $\omega_{k0}(t)$ in Eqs.(5.3), we obtain after equivalent reduction

$$2\,\mathrm{Re}\sum_{k=1}^{3} A_{nk}X_k(t_k) = 2\,\mathrm{Re}\sum_{k=4}^{6} B_{nk}\Theta_k(t_k) \quad (n = 1, 2, \ldots, 6) \tag{5.5}$$

$$t_k = \mathrm{Re}\, t + \mu_k \,\mathrm{Im}\, t, \; t \in l$$

Eqs.(5.5) represent a homogeneous boundary value problem of the type (3.3) for an unbounded anisotropic medium with a doubly-periodic system of piezoceramic inclusions.

By virtue of the uniqueness theorem we obtain

$$\begin{aligned} X_k(z_k) &= \Omega^* g_k^* z_k + d_k^* \quad (k = 1, 2, 3) \\ i\Theta_k(z_k) &= i(\Omega^* g_k^* z_k + d_k^*) \quad (k = 4, 5, 6) \end{aligned} \tag{5.6}$$

Comparing the functions $\Theta_k(z_k)$ from (5.6) and (5.2), we arrive at the conclusion that the expressions

$$Q_k(t) = \sum_{j=1}^{3} [l_{kj}\omega_{j0}(t) + l_{kj}^* \overline{\omega_{j0}(t)}] \quad (k = 4, 5, 6) \tag{5.7}$$

are boundary values of functions of the complex variable z_k that are regular in the domain $D_0^{(k)}$. Therefore, the representation

$$\Phi_{k0}(z_k) = \frac{1}{2\pi i}\int_l \frac{Q_k(t)\,dt_k}{t_k - z_k} \quad (k = 4, 5, 6); \quad z_k \in D_0^{(k)} \tag{5.8}$$

holds. We introduce the functions

$$i\Theta_k^*(z_k) = \frac{1}{2\pi i}\int_l \frac{Q_k(t)\,dt_k}{t_k - z_k} \quad (k = 4, 5, 6); \quad z_k \in D^{(k)} \tag{5.9}$$

The difference between the limit values of the functions (5.8) and (5.9) and l when Eqs. (5.1) and (5.4) are taken into account, gives

$$Q_k(t) = i\Theta_k^*(t_k) \; (k = 4, 5, 6) \tag{5.10}$$

Therefore, the second group of equations in (5.3) can be replaced by the relationships (5.10). Eliminating $\omega_{j0}(t)$ from these relationships exactly as before, we arrive at a homogeneous boundary value problem for an anisotropic medium with one piezoceramic inclusion

$$2\,\mathrm{Re}\sum_{k=1}^{3} A_{nk}X_k(t_k) = 2\,\mathrm{Re}\sum_{k=4}^{6} B_{nk}\Theta_k^*(t_k) \quad (n = 1, 2, \ldots, 6) \tag{5.11}$$

The functions (5.9) vanish at infinity; consequently, the boundary value problem (5.11) has just the trivial solution

$$\varkappa_k(z_k) = 0 \; (k = 1, 2, 3), \; \Theta_k^*(z_k) = 0 \; (k = 4, 5, 6) \tag{5.12}$$

Hence, by virtue of (5.3) and (5.4), we conclude that $\omega_{k0}(t) = 0$ $(k = 1, 2, 3)$, which it was required to prove.

6. Averaging of a piezoceramic structure. A determination of the macromodel of a regular structure is given in /11/. Expressing the mean strains and the mean values of the electric intensity vector components in terms of increments of the appropriate quantities by means of (2.1), using the quasiperiodicity of these functions, and the static and electrical conditions on the sides of the cells (2.2) and (2.3), we obtain the equations of state of the macromodel (we omit the angular brackets for brevity)

$$\begin{aligned} \varepsilon_x &= S_{11}\sigma_x + S_{13}\sigma_z + S_{15}\tau_{xz} + d_{11}E_x + d_{31}E_z \\ \varepsilon_z &= S_{31}\sigma_x + S_{33}\sigma_z + S_{35}\tau_{xz} + d_{13}E_x + d_{33}E_z \\ \gamma_{xz} &= S_{51}\sigma_x + S_{53}\sigma_z + S_{55}\tau_{xz} + d_{15}E_x + d_{35}E_z \\ D_x &= d_{11}\sigma_x + d_{13}\sigma_z + d_{15}\tau_{xz} + \varepsilon_{11}E_x + \varepsilon_{13}E_z \\ D_z &= d_{31}\sigma_x + d_{33}\sigma_z + d_{35}\tau_{xz} + \varepsilon_{13}E_x + \varepsilon_{33}E_z \end{aligned}$$

The coefficients $\langle S_{ij}\rangle$, $\langle d_{ij}\rangle$, $\langle \varepsilon_{ij}\rangle$ will, respectively, be called the average compliance, piezomoduli, and permittivities. They are functionals determined by certain standard solutions of system (3.3). We will not write them here because of their complexity.

7. Results of calculations. As an example, we consider a composite material with the piezoceramic matrix PZT-5 /9/, bonded linearly by fibres of circular transverse cross-section of radius R whose centres form a square lattice with spacing $\omega_1 = 2$. The fibre material is a boron epoxy with the parameters $b_{11} = 2.5\cdot 10^{-12}\,\mathrm{m}^2/\mathrm{N}$, $b_{33} = 25\cdot 10^{-12}\,\mathrm{m}^2/\mathrm{N}$, $b_{13} = b_{31} = -0.625\cdot 10^{-12}\,\mathrm{m}^2/\mathrm{N}$,

$b_{55} = 66.67 \cdot 10^{-12}$ m^2/N, $b_{15} = b_{35} = 0$, $\varkappa_{13} = 0$, $\varkappa_{11} = \varkappa_{33} = 4 \cdot 8.85 \cdot 10^{-12}$ C^2/Nm^2.

Results of computing the average structure parameters are shown in Fig.2. The solid curves *1—4* describe the relative piezoelectric modulus $\langle d_{15}\rangle/d_{15}^*$ and the quantities $\langle S_{33}\rangle/S_{33}^*$, $\langle S_{11}\rangle/S_{11}^*$, $\langle S_{13}\rangle/S_{13}^*$, respectively (the elastic compliances $\langle S_{15}\rangle$, $\langle S_{35}\rangle$, the piezoelectric modulus $\langle d_{35}\rangle$ and the permittivity $\langle \varepsilon_{13}\rangle$ are zero). The dashed curves *1—3* are constructed for the relative permittivities $\langle \varepsilon_{11}\rangle/\varepsilon_{11}^*$, $\langle \varepsilon_{33}\rangle/\varepsilon_{33}^*$, and the quantity $\langle S_{55}\rangle/S_{55}^*$ (the relative piezoelectric moduli $\langle d_{31}\rangle/d_{31}^*$ and $\langle d_{33}\rangle/d_{33}^*$ equal unity). We have introduced the notation

$$S_{11}^* = S_{11} - S_{12}^2/S_{11}, \quad S_{13}^* = S_{13} - S_{13}S_{12}/S_{11}$$
$$S_{33}^* = S_{33} - S_{13}^2/S_{11}, \quad S_{55}^* = S_{44}; \quad d_{15}^* = d_{15}$$
$$d_{31}^* = d_{31} - d_{31}S_{12}/S_{11}, \quad d_{33}^* = d_{33} - d_{31}S_{13}/S_{11};$$
$$\varepsilon_{11}^* = \varepsilon_{11}, \quad \varepsilon_{33}^* = \varepsilon_{33} - d_{31}^2/S_{11}$$

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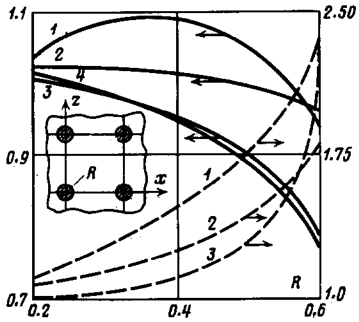

Fig.2

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